The Mathematics of Juggling

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ABSTRACT: This paper lays down the foundations for the mathematics of juggling starting with the continuity axiom and defining multi-hand notation, which describes a large class of juggling patterns. Several theorems are stated and proven using these definitions and the axiom of continuity. One need not know how to juggle to appreciate these ideas. The mathematics is interesting by itself and can be appreciated by the non-juggler as well as the juggler.

Introduction

Juggling is an art form which has been around for ages. There is much disagreement over the definition of juggling. Nowadays, the world of juggling includes cigar box manipulation, diablo, shaker cups, devil sticking, and even contact juggling, an art form popularized by Michael Moschen. For the purposes of this article, juggling shall be defined as keeping a number of objects in motion. This definition does not mention how objects are moved. In fact, it does not require any of the motions be periodic or even that hands be used to catch and throw the objects.

When there is repetition to the movements of objects, this is called a periodic pattern. The most widely known three-ball pattern is the cascade, in which the juggler alternates right- and left-hand throws in a regular rhythm making each throw the same height. There are thousands of patterns which involve alternating right- and left-hand throws in a regular rhythm. These patterns are called siteswaps and they lend themselves very nicely to a concise notation. (In siteswap notation, the three-ball cascade is simply 3.) Siteswap notation is the most widely known notation among jugglers today. It was invented by Paul Klymack from Santa Cruz. Siteswaps are briefly explained in an article, "The Physics of Juggling" by Bengt Magnusson and Bruce Tiemann (The Physics Teacher November 1989 p.586).

In the summer of 1991, I invented multi-hand notation (MHN), which is a concise and mathematical way of describing patterns involving any number of hands and not necessarily throwing in an alternating manner. This notation, which was originally presented to the usenet group, rec.juggling, is not yet as well-known as siteswap notation, however I believe it is indispensable as a tool for mathematical analysis of juggling. I also wrote a program called JugglePro which notates patterns in MHN and using high-resolution graphics, juggles balls on-screen according to the indicated pattern. This program is even capable of moving the hands while juggling and catching and throwing more than one ball at a time with one hand.

Representation of Pattern Space and Patterns

The *pattern space* in which juggling takes place is represented by (h,t) where h and t represent the hand and time at which a throw or catch occurs. In other words, throws and catches are events which take place at discrete points in (h,t) space. This is defined as

$$\wp \equiv \{(h,t): 1 \le h \le H \quad h,t \in Z\}$$

where H represents the number of hands and Z is the set of integers.

The juggler needs a set of instructions which describe where each ball is to be thrown. This is accomplished by means of a vector function, $\mathbf{p}_m$, which takes a point (h,t) and gives that throw's destination, $\mathbf{p}_m(h,t)$, which is another point in pattern space. The juggler may have more than one ball in one hand at a time. If there are M balls at (h,t), then the subscript m ranges from 1 to M so that there is a different function for each ball. The set of functions, $\{\mathbf{p}_m: \ 1 \le m \le M\}$, represents the *pattern*.

Equation of Continuity in Pattern Space

In this section, the simple notion of continuity is put into precise mathematical terms. To do this some definitions must be made first. When a ball is caught it is usually held for some time before it is thrown again. This time is called the *hold time*. In a mathematical treatment of a juggling pattern it is helpful to suppress the notion of hold time by letting it be zero. (There is no loss in generality in doing this.) A *frame* is any point in time where catches and throws occur (simultaneously). A *frametime* is the time interval between successive frames. For any particular frame and hand, the number of balls entering or being caught by the hand must equal the number of balls leaving or being thrown by that hand. This is the axiom of continuity. Expressed mathematically we have,

$$\sum_{\boldsymbol{x}\in\wp} \sum_{m=1}^{M} \delta(\boldsymbol{z}-\boldsymbol{p}_m(\boldsymbol{x})) = M \qquad (\forall \boldsymbol{z}\in\wp)$$

The equation tells us that if we choose any point, $\mathbf{z}$, in the pattern space, and we count the total number of throws, $\mathbf{p}_m(\mathbf{x})$ to that position, we get M, the number of throws being made from point $\mathbf{z}$. Patterns with M>1 have a special name given to them since they require the juggler to be able to catch and throw more than one ball using the same hand. They are called multiplex patterns.

Patterns are *valid* if they satisfy the equation of continuity. It is interesting to note, however, that a valid pattern is not necessarily physically realizable. A valid pattern can have throws which move the ball backwards in time! We will not deal with these rather abstract patterns until later. Therefore we add the jugglability condition - a pattern is *jugglable* if it is valid and has no throws backwards in time and if there is a throw with zero time component, it must be a self-throw. The latter statement takes care of the "rest" which occurs in a frame when a hand has no ball to catch.

Periodicity

The general equation of continuity is of little practical help because it requires an infinite summation. In practice, we encounter *periodic* patterns, which repeat after a certain number of frames. Therefore, we confine our interest to this class. Suppose a pattern repeats after L frames. We say its period is L and since it is periodic we have

$$\mathbf{r}_m(h,t) = \mathbf{r}_m(h,t+L)$$

where $\mathbf{r}_m(h,t)$ is the displacement vector representing the M throw(s) from position (h,t) and is given by

$$\mathbf{r}_m(h,t) = \mathbf{p}_m(h,t) - (h,t).$$

Given that the pattern is periodic, we can restrict (h,t) to one period of pattern space. It is sufficient to know only one period of the pattern in order to know the entire pattern. We define the period space to be

$$\mathfrak{L} \equiv \{(h,t): 1 \le h \le H,\ 1 \le t \le L\}$$

The equivalent definition of periodicity for the positional form of the pattern can now be found. If a pattern P is periodic with period L, then

$$\boldsymbol{p_m}(h,t) + k(0,L) = \boldsymbol{p_m}(h,t+kL) \quad \forall k \in Z$$

The pattern can be represented by an HxLxM matrix of vectors, $\mathbf{p}_m(h,t)$. This brings us to multi-hand notation (MHN), which is devised to represent periodic patterns.

Multi-Hand Notation (MHN)

A throw takes an object (a ball) to a new location in (h,t) space. This destination vector is placed at the location in (h,t) space from which the throw initiates. A matrix of vectors describes the throws making up the pattern. This is the *positional form* of the pattern. The subscript m is used for multiplex patterns. For M>1, a different 2-d matrix would be needed for each m=1,2,...M. Thus, the multiplex pattern matrix becomes three dimensional.

$$P = \begin{bmatrix} \boldsymbol{p}_m(1,1) & \boldsymbol{p}_m(1,2) & \cdots & \boldsymbol{p}_m(1,L) \\ \boldsymbol{p}_m(2,1) & & & \\ \cdot & & & \\ \cdot & & & \\ \boldsymbol{p}_m(H,1) & \boldsymbol{p}_m(H,2) & \cdots & \boldsymbol{p}_m(H,L) \end{bmatrix}$$

The zero permutation matrix, P_0, takes each (h,t) location to itself.

$$P_0 = \begin{bmatrix} (1,1) & (1,2) & \cdots & (1,L) \\ (2,1) & & & \\ \cdot & & & \\ \cdot & & & \\ (H,1) & (H,2) & \cdots & (H,L) \end{bmatrix}$$

The relative form of a pattern is found by subtracting P_0 from P.

$$R \equiv P - P_0$$

As we shall see later, P_0, is the identity for composition of permutations.

Equation of Continuity for Periodic Patterns

In this section we develop a method of verifying continuity for a periodic pattern which involves a summation over just one period. Again the concept is to count the number of balls being thrown to a particular hand in a frame and check to see if this count equals M, the number of balls being thrown from the hand. If we restrict our count to just one period, we will miss those balls whose throws originate outside this period. How do we account for these? Consider a throw that originates in the previous period. The concept of periodicity tells us that the same throw takes place exactly L frametimes later, which is in the period of interest. Therefore, we count this as one of the balls being thrown to the hand of interest.

Theorem 1

$$\sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} \delta(\boldsymbol{z}-\boldsymbol{p}_m'(\boldsymbol{x})) = M \qquad \forall \boldsymbol{z}\in\wp$$

where we define $\mathbf{p}_m'(\mathbf{x})$ as follows:

Let $(h_p, t_p) = \mathbf{p}_m(\mathbf{x})$ and let $(h_z, t_z) = \mathbf{z}$.

$$\boldsymbol{p}_m'(\boldsymbol{x}) \equiv \boldsymbol{p}_m(\boldsymbol{x}) - (0, [\![(t_p - t_z)/L]\!] \cdot L)$$

Proof

We start with the general equation of continuity and sum it period by period.

$$\sum_{\tau=-\infty}^{\infty} \sum_{t=1-\tau L}^{L-\tau L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{z}-\boldsymbol{p}_m(h,t)) = M$$

Applying the definition of periodicity to the equation of continuity, we obtain:

$$\sum_{\tau=-\infty}^{\infty} \sum_{t=1-\tau L}^{L-\tau L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{z} - \boldsymbol{p}_m(h, t+\tau L) + (0,\tau L)) = M$$

Now we change the order of the sum and let $t' = t + \tau L$.

$$\sum_{t'=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{\tau=-\infty}^{\infty} \delta(\boldsymbol{z} - \boldsymbol{p}_m(h, t') + (0,\tau L)) = M$$

There is but one integer τ which can possibly satisfy:

$$\boldsymbol{z} = \boldsymbol{p}_m(h, t') - (0, \tau L).$$

It is

$$\tau' = [\![(t_p - t_z)/L]\!]$$

Thus, we can drop the summation over τ by replacing $\mathbf{p}_m$ by $\mathbf{p}_m'$ as defined in theorem 1. When we do this, we get equation of Theorem 1.

Multiplex Space

So far, we have been denoting throw-vectors by $\mathbf{p}_m(h,t)$. The subscript m can take on integer values from 1 to M, the multiplex limit. The multiplex limit is the number of balls which can be caught and thrown from one hand at one time. We have been considering these M different throws occuring at (h,t) as being interchangeable. From now on, however, we will identify each throw with a number from 1 to M. We begin by defining *multiplex space*.

$$\mathrm{M} \equiv \{(m,h,t) : 1 \le m \le M,\ 1 \le h \le H,\ -\infty < t < \infty\}$$

We can think of a single vector function, **p**, mapping multiplex space to pattern space. This function is not one-to-one because there are M vectors, **x** = (m,h,t), which all map to the same position for each position in pattern space. We would like to have a one-to-one function. We proceed by adding one more dimension to the range of **p**, making it be a one-to-one mapping from multiplex space to multiplex space.

For each **z** = (h,t) in pattern space, there are M vectors **x** in multiplex space which map to **z** under the function **p**. Denote these vectors $\mathbf{x}_m$ where m goes from 1 to M. Redefine **z** = (m,h,t). Now **p** maps multiplex space to multiplex space. Each **x** is mapped to one **z** by the function **p** because **p** is single-valued. For each **z** = (m,h,t), there exists but one **x** for which **p**(**x**) = **z**. Therefore, **p** is a bijection.

For multiplex space, the equation of continuity becomes

$$\sum_{\boldsymbol{x} \in \mathrm{M}} \delta(\boldsymbol{p}(\boldsymbol{x}) - \boldsymbol{z}) = 1 \quad \forall \boldsymbol{z} \in \mathrm{M}.$$

When restricted to a period, the equation can be stated as

$$\sum_{\boldsymbol{x} \in \mathrm{M}'} \delta(\boldsymbol{p}'(\boldsymbol{x}) - \boldsymbol{z}) = 1 \quad \forall \boldsymbol{z} \in \mathrm{M}'.$$

where

$$\mathrm{M}' \equiv \{ (m,h,t) : 1 \le m \le M,\ 1 \le h \le H,\ 1 \le t \le L \}$$

and where both the domain and range of **p′** is restricted to one period of multiplex space in such a way that

$$\boldsymbol{p}'(\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x}) - (0,0,\tau(\boldsymbol{x})\cdot L) \quad \textit{where } \tau(\boldsymbol{x}) \textit{ is an integer}.$$

The equation of continuity shows that **p′** is a bijection and therefore it is a permutation on restricted multiplex space. We have proven a valuable theorem about periodic patterns. It allows one to quickly generate hosts of valid patterns by simple permutations and additions.

Theorem 2.

$$P = perm(P_0^{/}) + (0,0,L)\cdot E.$$

where E is the excitation matrix of integers

$$E = \begin{bmatrix} \tau(m,1,1) & \tau(m,1,2) & \ldots & \tau(m,1,L) \\ \tau(m,2,1) & \tau(m,2,2) & \ldots & \tau(m,2,L) \\ . & & & \\ . & & & \\ \tau(m,H,1) & \tau(m,H,2) & \ldots & \tau(m,H,L) \end{bmatrix}.$$

E is really a three-dimensional matrix whose elements, $\tau(\mathbf{x})$, are the integers used in defining $\mathbf{p}'(\mathbf{x})$. Theorem 2 defines **p** only in restricted multiplex space. The domain of **p** can be extended to multiplex space by using the fact that **p** is periodic. Any element of multiplex space can be written as ($\mathbf{x}$ + k(0,0,L)) where **x** is an element of restricted multiplex space and k is an integer.

$$\mathbf{p}(\mathbf{x} + k(0,0,L)) = \mathbf{p}(\mathbf{x}) + k(0,0,L) = \mathbf{p}'(\mathbf{x}) + (0,0,L)\cdot(\tau(\mathbf{x})+k).$$

Theorem 2 can be stated in words: Any valid periodic pattern is representable by the sum of some permutation matrix and (0,0,L) times some excitation matrix.

Pattern Operations which Preserve Validity

There are three useful operations which can be easily proven to preserve validity. They are permutation, local translation, and global translation. Consider an arbitrary permutation on restricted multiplex space, $\mathbf{q}(\mathbf{x})$. Suppose that **p** is a valid pattern. $\mathbf{q}(\mathbf{x})$ is in restricted multiplex space so Theorem 2 says:

$$\mathbf{p}(\mathbf{q}(\mathbf{x})) = \mathbf{p}'(\mathbf{q}(\mathbf{x})) + (0,0,L)\cdot\tau(\mathbf{q}(\mathbf{x}))$$

$\mathbf{p}'(\mathbf{q})$ is a permutation on restricted multiplex space and $\tau(\mathbf{q}(\mathbf{x}))$ is defined and is an integer. As done before, we extend the domain of $\mathbf{p}(\mathbf{q})$ to multiplex space by using the fact that it is periodic and this gives a valid pattern.

A local translation is defined as an addition of any integer number times L to any one of the time components of the throw-vectors in the matrix. It is also easily proven from Theorem 2.

$$P + (0,0,L)\cdot E_{local} = perm(P_0^{/}) + (0,0,L)\cdot(E_{local} + E)$$

The new excitation matrix for $(P + (0,0,L)E_{local})$ is $(E + E_{local})$. Hence, the local translation of P is also valid.

A global translation is defined as the addition of some integer, g, to the time components of all of the vectors in the matrix. It is proven using the general equation of continuity.

$$\sum_{\boldsymbol{x}\in M} \delta(\boldsymbol{p}(\boldsymbol{x}) - [\boldsymbol{z}-(0,0,g)]) = 1 \qquad \forall \boldsymbol{z}\in M.$$

$$\sum_{\boldsymbol{x}\in M} \delta([\boldsymbol{p}(\boldsymbol{x})+(0,0,g)]-\boldsymbol{z}) = 1 \qquad \forall \boldsymbol{z}\in M.$$

The elements of the globally translated matrix are $\mathbf{p}(\mathbf{x})+(0,0,g)$. Therefore, global translation preserves validity.

States and Transitions

So far we have only looked at patterns by themselves. That is, we have not thought about what a juggler might do to go from pattern X to pattern Y. Our present notation does not tell us where all the balls are at every frame of the pattern. It only tells us what throws to make at each frame. To denote the configuration of balls in (h,t) space we define a *state matrix*. It should be noted that the state matrix as defined here is only for those valid patterns whose throws are all forward into time.

$$S_0 = \begin{bmatrix} s(1,1) & s(1,2) & \dots & s(1,W) \\ s(2,1) & s(2,2) & \dots & s(2,W) \\ \cdot & & & \\ s(H,1) & s(H,2) & \dots & s(H,W) \end{bmatrix}$$

$$W \equiv \max t_r \quad \textit{where we let } (h_r, t_r) = r_m(x) \qquad (x\in\mathfrak{L})$$

This matrix is HxW where W, the *width* of the matrix, is the maximum throw height. Each element tells how many balls there are at that location in (h,t) space. Elements beyond this width, W, are all taken to be zero. This *state space* is a subset of pattern space and is defined

$$\sigma = \{(h,t): \ 1\le h\le H, \ \ 0\le t<\infty\}$$

As time passes, each ball moves to the left in the matrix. To be precise, after one frametime each element gets shifted one column to the left. Then each hand makes new throws prescribed by the pattern matrix and the state matrix becomes

$$S_1 = \begin{bmatrix} s(1,2) & s(1,3) & \ldots & s(1,W) & 0 \\ s(2,2) & s(2,3) & \ldots & s(2,W) & 0 \\ . & & & & \\ s(H,2) & s(H,3) & \ldots & s(H,W) & 0 \end{bmatrix} + S_1'$$

where the components of S'_t are defined as

$$s_t'(i,j) \equiv \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,t) - (0,t) - (i,j))$$

This is what is called a *transition*. The fundamental transition equation is

$$S_{a+1} = D(S_a) + S_{a+1}'$$

where a is any integer and $D(S_a)$ is the decay of S_a by one frametime (i.e. left-shift). Expressed in terms of components we have

$$s_{a+1}(i,j) = s_a(i,j+1) + \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,a+1) - (0,a+1) - (i,j))$$

Theorem 3
For any integer $t \geq 1$,

$$s_{a+t}(i,j) = s_a(i,j+t) + \sum_{l=a+1}^{a+t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+a+t))$$

Proof
From the fundamental transition equation, we have inductively that

$$S_{a+t} = D^{(t)}(S_a) + D^{(t-1)}(S_{a+1}') + D^{(t-2)}(S_{a+2}') + \ldots + S_{a+t}'$$

Stating this equation in terms of components we have

$$s_{a+t}(i,j) = s_a(i,j+t) + \sum_{h=1}^{H} \sum_{m=1}^{M} [\ \delta(\boldsymbol{p}_m(h,a+1)-(0,a+1)-(i,j+t-1))$$

$$+ \delta(\boldsymbol{p}_m(h,a+2)-(0,a+2)-(i,j+t-2)) + \ldots$$

$$+ \delta(\boldsymbol{p}_m(h,a+t)-(0,a+t)-(i,j))\ ]$$

These sums can now be combined under a third summation. When this is done, we get Theorem 3.

Corollary 3A

For any integer $t \geq W$,

$$s_{a+t}(i,j) = \sum_{l=a+1}^{a+t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+a+t))$$

Proof
This is obvious because $s_a(i,j+t) = 0$ for $t \geq W$.

A periodic pattern of period L goes through L states. In the ith frame, the throws would be given by the ith column of the pattern matrix. The state matrix for the ith frame would be S_i. One might say at first that there is a flaw in this definition because it requires that you first know the previous state. However, it is clear from Corollary 3A that after $t \geq W$ frames all of the elements of S_a have been shifted out so that S_{a+t} no longer depends on S_a but only on the pattern matrix, P. Furthermore, if the pattern matrix, P, is completely known, then any state in the pattern can be found directly from P. This must be true since a can be chosen arbitrarily.

The periodicity of the pattern implies that states must repeat after L frames. We prove this theorem next.

Theorem 4

$$S_t = S_{(t+L)}$$

Proof
We start with a statement of Corollary 3A.

$$s_{t+L}(i,j) = \sum_{l=t+L-W}^{t+L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l)-(i,j+t+L))$$

From the definition of periodicity we get,

$$= \sum_{l=t+L-W}^{t+L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h, l-L) - (i, j+t))$$

$$= \sum_{l=t-W}^{t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(p_m(h, l) - (i, j+t)) = s_t(i, j) .$$

Theorem 5
For a valid pattern (one which satisfies the continuity equation) the components of S are bounded by M.

Proof
From Corollary 3A we have

$$s_{a+t}(i, j) = \sum_{l=a+1}^{a+t} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h, l) - (i, j+a+t)) \quad \textit{for } t \geq W$$

From the general equation of continuity we have

$$M = \sum_{l=-\infty}^{\infty} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h, l) - (i, j+a+t)) \geq s_{a+t}(i, j)$$

Intuitively, we know that a component of S must not exceed the multiplex limit, M. M represents the total number of balls one hand can handle at one time. If s(i,j) were to exceed M, then when it came time to catch the s(i,j) balls, there would be too many to handle.

The following theorem is an extension of Shannon's Theorem. It relates the number of balls, N, to the period, L, and the relative pattern matrix, R.

Theorem 6

$$\sum_{\boldsymbol{x} \in M'} \boldsymbol{r}(\boldsymbol{x}) = (0, 0, NL)$$

where

$$N = \sum_{\boldsymbol{x} \in \sigma} s_a(\boldsymbol{x}) \qquad \forall a.$$

Proof
From Theorem 2 we have,

$$P = perm(P_0') + (0,0,L) \cdot E.$$

where P is the pattern matrix of vectors (m,h,t).

Summing over restricted multiplex space and applying the commutative property of addition to perm(P'_0),

$$\sum_{\boldsymbol{x} \in M'} \boldsymbol{p}(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in M'} \boldsymbol{p}_0'(\boldsymbol{x}) + \sum_{\boldsymbol{x} \in M'} (0,0,L) \cdot \tau(\boldsymbol{x})$$

Since $\mathbf{p}'_0(\mathbf{x}) = \mathbf{x}$,

$$\sum_{\boldsymbol{x} \in M} (\boldsymbol{p}(\boldsymbol{x}) - \boldsymbol{x}) = \sum_{\boldsymbol{x} \in M} \boldsymbol{r}(\boldsymbol{x}) = (0,0,L) \cdot \sum_{\boldsymbol{x} \in M} \tau(\boldsymbol{x}).$$

This proves the theorem for the first two components. I have not found a simple proof for the last component. We start with the transition theorem for a=0 and t=L. ($\mathbf{p}_m$ maps pattern space into pattern space.)

$$s_L(i,j) = s_0(i,j+L) + \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(h,l) - (i,j+L))$$

Now we multiply both sides by j and form a sum over state space.

$$\sum_{i=1}^{H} \sum_{j=1}^{\infty} j s_L(i,j) = \sum_{i=1}^{H} \sum_{j=1}^{\infty} j s_0(i,j+L) + term2$$

$$term2 = \sum_{i=1}^{H} \sum_{j=1}^{\infty} j \sum_{\boldsymbol{x} \in \mathcal{L}} \sum_{m=1}^{M} \delta(\boldsymbol{p}_m(\boldsymbol{x}) - (i,j+L))$$

$$term1 = \sum_{h=1}^{H} \sum_{j=1}^{\infty} j \; s_0(i,j+L) = \sum_{h=1}^{H} \sum_{j=1}^{\infty} (j+L-L) \; s_0(i,j+L)$$

$$= \sum_{i=1}^{H} \sum_{j=1+L}^{\infty} j\ s_0(i,j) - \sum_{i=1}^{H} \sum_{j=1+L}^{\infty} L\ s_0(i,j)$$

$$= \left[\sum_{i=1}^{H} \sum_{j=1}^{\infty} j\ s_0(i,j) - \sum_{i=1}^{H} \sum_{j=1}^{L} j\ s_0(i,j) \right]$$

$$- L \left[\sum_{i=1}^{H} \sum_{j=1}^{\infty} s_0(i,j) - \sum_{i=1}^{H} \sum_{j=1}^{L} s_0(i,j) \right]$$

$$= \left[\sum_{h=1}^{H} \sum_{j=1}^{\infty} j\ s_0(i,j) \right] - L{\cdot}N + \sum_{i=1}^{H} \sum_{j=1}^{L} (L-j)\ s_0(i,j).$$

We now define the *excitation value*, XV, of a state, S.

$$XV(S) \equiv \sum_{i=1}^{H} \sum_{j=1}^{\infty} j\ s(i,j).$$

Our total equation now becomes

$$XV(S_L) = XV(S_0) - LN + \sum_{i=1}^{H} \sum_{j=1}^{L} (L-j) s_0(i,j) + term2$$

We know that since $S_0=S_L$, $XV(S_0)=XV(S_L)$. Thus, the equation becomes

$$NL = \sum_{j=1}^{L} \sum_{i=1}^{H} (L-j) s_0(i,j) + \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \sum_{j=1}^{\infty} j\ \delta(p_m(h,l) - (i,j+L)).$$

$$term2 = \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \sum_{j=1}^{\infty} j\delta(\boldsymbol{p_m}(h,l) - (i,j+L))$$

Collecting terms we have,

$$= \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \left[\sum_{j=-\infty}^{\infty} j\delta(\boldsymbol{p_m}(h,l) - (i,j+L)) - \sum_{j=-\infty}^{0} j\delta(\boldsymbol{p_m}(h,l) - (i,j+L)) \right]$$

$$= \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (t_p - L) - \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \sum_{i=1}^{H} \sum_{j=-\infty}^{L} (j - L)\delta(\boldsymbol{p_m}(h,l) - (i,j))$$

$$= \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (t_R + l - L) + \sum_{j=1}^{L} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\delta(\boldsymbol{p_m}(h,l) - (i,j))$$

$$+ \sum_{j=-\infty}^{0} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\delta(\boldsymbol{p_m}(h,l) - (i,j))$$

$$\sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (t_R(h,l) + l - L) = \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) + \sum_{j=1}^{L} \sum_{i=1}^{H} \sum_{m=1}^{M} (j - L).$$

$$NL = \sum_{j=1}^{L} \sum_{i=1}^{H} (L - j)\left[s_0(i,j) - M + \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h,l) - (i,j)) \right]$$

$$+ \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) + \sum_{j=-\infty}^{0} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\delta(\boldsymbol{p_m}(h,l) - (i,j))$$

For the first term, we make the following substitutions for $s_0(i,j)$ and M:

$$s_0(i,j) = \sum_{l=-\infty}^{0} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h,l) - (i,j)) \quad (\textit{Corollary 2A})$$

$$M = \sum_{l=-\infty}^{\infty} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h,l) - (i,j)) \quad (\textit{continuity})$$

The equation reduces to

$$NL = \sum_{\boldsymbol{x}\in\mathfrak{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) - \sum_{j=1}^{L} \sum_{i=1}^{H} (L - j) \sum_{l=L+1}^{\infty} \sum_{h=1}^{H} \sum_{m=1}^{M} \delta(\boldsymbol{p_m}(h,l) - (i,j))$$

$$+ \sum_{j=-\infty}^{0} \sum_{i=1}^{H} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{M} (L - j)\,\boldsymbol{\delta}(\boldsymbol{p_m}(h,l) - (i,j))$$

The last two terms are both identically zero. The reason is that throws cannot be made backwards in time. This means

$$t_R(\boldsymbol{x}) \geq 0 \quad \Rightarrow \quad t_p(\boldsymbol{x}) \geq t_x$$

where $\mathbf{p}_m(\mathbf{x}) = (h_p(\mathbf{x}), t_p(\mathbf{x}))$,

$\mathbf{r}_m(\mathbf{x}) = (h_R(\mathbf{x}), t_R(\mathbf{x}))$,

and $\mathbf{x} = (h_x, t_x)$.

In the second to last sum, the condition of jugglability implies $t_p(h,l) \geq l$. Clearly, $j<l$ in this sum. Hence, $j<t_p(h,l)$. Since j never equals $t_p(h,l)$, the total sum is zero. Likewise, in the very last sum, $j<l \leq t_p(h,l)$. Therefore its total is also zero. That leaves the equation,

$$NL = \sum_{\boldsymbol{x} \in \mathcal{L}} \sum_{m=1}^{M} t_R(\boldsymbol{x}) \,.$$

This concludes the proof of Theorem 6. As a corollary, we add the obvious conclusion that the number of balls, N, is equal to the sum of elements of the excited state matrix, E.

Corollary 6A

$$N \equiv \sum_{\boldsymbol{x} \in \sigma} s_a(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in M'} \tau(\boldsymbol{x}) \,.$$

Partitioning Multiplex Space into Paths

In this section, we will be using some concepts from abstract algebra. We will see that multiplex space can be partitioned into paths. A path can be thought of as the path of a ball in multiplex space. One might guess that the number of paths determines the number of balls. This is true, but as we will find out later, some of these balls may not be real! In fact, if we allow patterns with throws that are backwards in time, these patterns may involve "antiballs".

Theorem 7
The set of valid patterns together with the operation of composition forms a group.

Proof
The identity is $\mathbf{p}_0(\mathbf{x})=\mathbf{x}$ which maps every element of multiplex space to itself. Composition of functions is an associative operation. Every valid pattern, $\mathbf{p}$, has an inverse because it is a bijection on multiplex space (see section on Multiplex Space).

Definitions:

We define the *positive* $\mathbf{p}$ *path of* $\mathbf{x}$ to be the generated set of $\mathbf{p}$.

$$Path_p^+(\boldsymbol{x}) \equiv \{ \boldsymbol{p}^{\boldsymbol{k}}(\boldsymbol{x}) : \ k \in \mathrm{N} \}$$

We define the *negative* $\mathbf{p}$ *path of* $\mathbf{x}$ to be the generated set of $\mathbf{p}^{-1}$.

$$Path_p^-(\boldsymbol{x}) \equiv \{ \boldsymbol{p}^{-\boldsymbol{k}}(\boldsymbol{x}) : \ k \in \mathrm{N} \}$$

A positive or negative path *diverges to positive infinity* if given any B>0, there is a positive integer T such that when j≥T, the time component of $\mathbf{p}^j(\mathbf{x})$ is greater than B. Similarly, a path *diverges to negative infinity* if given any A<0, there is a negative integer T such that when j≤T, the time component of $\mathbf{p}^j(\mathbf{x})$ is less than A.

Consider $\mathbf{p}'$ to be the permutation on restricted multiplex space (M′) as defined in theorem 2. From abstract algebra, we know that $\mathbf{p}'$ can be partitioned into disjoint cycles. This means that for any member $\mathbf{x}$ of M′, $\mathbf{x}$ is moved by a cycle or is a fixed point. In other words, there exists an integer, l, such that $\mathbf{p}'^{l}(\mathbf{x}) = \mathbf{x}$.

We define the *orbit* to be

$$O_x \equiv \{ \boldsymbol{p}'^{j}(\boldsymbol{x}) : \ 0 \le j < l, \ j \in Z \} \ \textit{where } \boldsymbol{p}'^{l}(\boldsymbol{x}) = \boldsymbol{x}.$$

Theorem 8
For a periodic valid pattern, **p**, and for **x** in multiplex space, there exists an integer, l≥1 which satisfies:

$$\mathbf{p}^{l}(\mathbf{x}) = \mathbf{x} + (0,0,L)\cdot\sum_{\mathbf{z}\in O_x} \tau(\mathbf{x})$$

Proof
From theorem 2,

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}'(\mathbf{x}) + (0,0,L)\cdot\tau(\mathbf{x})$$

Using periodicity,

$$\mathbf{p}^2(\mathbf{x}) = \mathbf{p}(\mathbf{p}'(\mathbf{x}) + (0,0,L)\cdot\tau(\mathbf{x})) = \mathbf{p}(\mathbf{p}'(\mathbf{x})) + (0,0,L)\cdot\tau(\mathbf{x})$$

so

$$\mathbf{p}^2(\mathbf{x}) = \mathbf{p}'^2(\mathbf{x}) + (0,0,L)\cdot(\tau(\mathbf{p}'(\mathbf{x})) + \tau(\mathbf{x})).$$

Proceeding inductively, we find

$$\mathbf{p}^{l}(\mathbf{x}) = \mathbf{p}'^{l}(\mathbf{x}) + (0,0,L)\cdot\sum_{\mathbf{z}\in O_x} \tau(\mathbf{z})$$

where l is the number of elements in the orbit of **x**. Hence, $\mathbf{p}'^{l}(\mathbf{x}) = \mathbf{x}$, and this yields theorem 8.

Definition: We refer to the summation of τ(**x**) over the orbit as the *gain* (K) of the positive **p** path of **x**.

Corollary 8A
The positive **p** path of **x** diverges to positive infinity if and only if the negative **p** path of **x** diverges to negative infinity.
Similarly, the positive **p** path of **x** diverges to negative infinity if and only if the negative **p** path of **x** diverges to positive infinity.

Proof
Let K = the gain of the positive **p** path of **x**. It is easily shown by induction on theorem 8 that $\mathbf{p}^{jl}(\mathbf{x}) = \mathbf{x} + jK\cdot(0,0,L)$. Let $\mathbf{x} = (m,h,t)$, $\mathbf{p}^{jl}(\mathbf{x}) = (m_p,h_p,t_p)$, and choose arbitrary bounds A and B. Notice that $t_p = t + jKL$.

If K>0 then choose positive j such that $j > (B-t)/KL$. This implies $t_p > B$. $\mathbf{p}^{jl}(\mathbf{x})$ is in the positive **p** path of **x**. Hence, the positive **p** path of **x** diverges to positive infinity. Now choose negative j such that $j < (A-t)/KL$. This implies $t_p < A$. Hence, the negative **p** path of **x** diverges to negative infinity.

If $K<0$ then choose positive j such that $j > (A-t)/KL$. This implies $jKL < (A-t)$ since K is negative. Hence $t_p < A$ and the positive path diverges to negative infinity. For negative j such that $j < (B-t)/KL$, we get $jKL > (B-t)$, so $t_p > B$ and the negative path diverges to positive infinity.

A path with zero gain is simply an orbit with 1 elements.

Definitions: We define the **p** *path of* **x** to be the union of positive and negative **p** paths of **x** together with **x** itself. The **p** path is *forward* if the positive **p** path diverges to positive infinity. It is *backward* if the positive **p** path diverges to negative infinity. The *gain* of the **p** path equals the gain of its positive **p** path. Given these definitions, it is clear from the proof above that paths with positive gains are forward and paths with negative gains are backward.

Theorem 9
The following statements about a **p** path are equivalent:

(i) A **p** path is an orbit.
(ii) The positive and negative **p** paths do not diverge.
(iii) Its gain is zero.

Proof
We have noted above that **p**'s path forms an orbit if and only if it has zero gain so (i) is equivalent to (iii). We now proceed to show (i) is equivalent to (ii).

Let **x** be some point in multiplex space. Suppose the **p** path of **x** is an orbit. Then the set of elements (m,h,t) of this path is finite. Let B be the maximum value of t for all such elements. Then there is no $t>B$. This implies the positive and negative **p** paths do not diverge to positive infinity. Now let B be the minimum value of t for all such elements. Then there is no $t<B$. This implies the positive and negative **p** paths do not diverge to negative infinity.

Now suppose the positive and negative **p** paths of **x** do not diverge. Let the elements of these paths by denoted (m,h,t). This means there is an upper bound, B, such that no t is greater than B. There is also a lower bound, A, such that no t is less than A. There are (B–A+1)*H*M elements of multiplex space meeting these conditions. Since this number is finite, the path must eventually become cyclic with an orbit having no greater than (B–A+1)*H*M elements.

Theorem 10
The set of **p** paths in multiplex space forms a partition.

Proof
We must show that distinct **p** paths are disjoint and that their union equals multiplex space. Consider the **p** path of **x** and the **p** path of **y**. Suppose that both paths contain the element, **z**. Then there is some integer k such that $\mathbf{z} = \mathbf{p}^k(\mathbf{x})$ and some integer j such that $\mathbf{z} = \mathbf{p}^j(\mathbf{y})$. Solving for **x** gives $\mathbf{x} = \mathbf{p}^{j-k}(\mathbf{y})$ which implies that for any integer, i, $\mathbf{p}^i(\mathbf{x}) = \mathbf{p}^{i+j-k}(\mathbf{y})$. Thus every element of the **p** path of **x** is in the **p** path of **y**. Similarly, we can show that every element of the **p** path of **y** is in the **p** path of **x**. This means the

paths are equal. We have shown two paths are equal if they share an element. The contrapositive of this is that two distinct **p** paths are disjoint. Next, consider a point **z** in multiplex space. **z** is a member of its own **p** path so it is a member of the union of distinct **p** paths. Hence, the union of distinct **p** paths equals multiplex space.

We can partition paths into "spatially distinct" paths. In theorem 8 it was proved that

$$\mathbf{p}^j(\mathbf{x}) = \mathbf{p}'^j(\mathbf{x}) + k_1\cdot(0,0,L) \qquad (\mathbf{x} \text{ in } M' \text{ and } k_1 \text{ in } Z).$$

Consider any **x'** in multiplex space. There is some integer, k_2 such that $\mathbf{x}' = \mathbf{x} + k_2\cdot(0,0,L)$. By periodicity,

$$\mathbf{p}^j(\mathbf{x}') = \mathbf{p}^j(\mathbf{x} + k_2\cdot(0,0,L)) = \mathbf{p}'^j(\mathbf{x}) + (k_1 + k_2)\cdot(0,0,L).$$

$$\mathbf{p}^j(\mathbf{x}') - \mathbf{p}'^j(\mathbf{x}) = k\cdot(0,0,L).$$

We say that the elements, $\mathbf{p}'^j(\mathbf{x})$, form an orbit which corresponds with the **p** path of **x'**. *Spatial equivalence* means equal orbits.

$$\text{Path}_p(\mathbf{x}') \sim \text{Path}_p(\mathbf{y}') \text{ if and only if } O_x = O_y$$

where O_x corresponds with the **p** path of **x'** and O_y, with the **p** path of **y'**. The paths do not necessarily have any elements in M' but they still correspond to disjoint orbits in M'.

Obviously, this defines an equivalence relation because it satisfies reflexive, symmetric, and transitive properties. We partition paths into spatial equivalence classes.

Theorem 11
$\text{Path}_p(\mathbf{x}') \sim \text{Path}_p(\mathbf{y}')$ if and only if there exists an element **z** of $\text{Path}_p(\mathbf{y}')$ such that $\mathbf{x}' = \mathbf{z} + k\cdot(0,0,L)$ where k is an integer.

Proof
Let **x** be the element of O_x, the corresponding orbit of $\text{Path}_p(\mathbf{x}')$, such that $\mathbf{x}' - \mathbf{x} = k_1\cdot(0,0,L)$. Since the path of **x'** is spatially equivalent to the path of **y'** their orbits are equal (i.e. $O_x = O_y$). Hence, there is an integer i such that $\mathbf{p}'_i(\mathbf{y}) = \mathbf{x}$. O_y is a corresponding orbit of $\text{Path}_p(\mathbf{y}')$ implies $\mathbf{p}_i(\mathbf{y}') - \mathbf{p}'_i(\mathbf{y}) = k_2\cdot(0,0,L)$.

$$\mathbf{x}' - \mathbf{x} = k_1\cdot(0,0,L)$$
$$\mathbf{p}_i(\mathbf{y}') - \mathbf{p}'_i(\mathbf{y}) = k_2\cdot(0,0,L)$$

Subtracting these gives

$$\mathbf{x}' - \mathbf{p}_i(\mathbf{y}') = k\cdot(0,0,L).$$

We have found $\mathbf{z} = \mathbf{p}_i(\mathbf{y}')$.

Now suppose we now $\mathbf{x}' = \mathbf{p}_i(\mathbf{y}') + k\cdot(0,0,L)$. Let O_x and O_y be the corresponding orbits of the paths of **x'** and **y'** respectively. Then $\mathbf{p}_i(\mathbf{y}') = \mathbf{p}'_i(\mathbf{y}) + k_2\cdot(0,0,L)$. Substituting, we find $\mathbf{x}' = \mathbf{p}'_i(\mathbf{y}) + (k+k_2)\cdot(0,0,L)$. $\mathbf{x}' = \mathbf{x} + k_1\cdot(0,0,L)$. Substituting this, we get $\mathbf{x} = \mathbf{p}'_i(\mathbf{y}) + (k+k_2-k_1)\cdot(0,0,L)$. Since **x** and $\mathbf{p}'_i(\mathbf{y})$ are in M', $k+k_2-k_1 = 0$, and $\mathbf{x} = \mathbf{p}'_i(\mathbf{y})$. Therefore, $O_x = O_y$ which means the two paths are spatially equivalent.

Theorem 12
The number of forward **p** paths minus the number of backward **p** paths is equal to the summation of $\tau(\mathbf{x})$ over restricted multiplex space (M′).

Proof
We first proof the following lemma.

Lemma: If K is the gain of a path, $Path_p(\mathbf{x}')$, and K>0 then it belongs to an equivalence class with K members. If K<0 then the equivalence class has −K members. For both cases, these members are $Path_p(\mathbf{x}')$, $Path_p(\mathbf{x}'+(0,0,L))$, $Path_p(\mathbf{x}'+(0,0,2L))$, .. $Path_p(\mathbf{x}'+(0,0,|K|L-L))$.

We must show these members are spatially equivalent and that these represent all paths which are spatially equivalent. By theorem 11, $Path_p(\mathbf{x}') \sim Path_p(\mathbf{x}' + j\cdot(0,0,L))$ for any integer j. So all members are spatially equivalent.

$$Path_p(\boldsymbol{x}) = \{\, \boldsymbol{p}^j(\boldsymbol{x}) : \; j \in Z \,\} = Path_p(\boldsymbol{p}^1(\boldsymbol{x})) = Path_p(\boldsymbol{x}+K\cdot(0,0,L)) .$$

Now consider a spatially equivalent path, $Path_p(\mathbf{x}'+j\cdot(0,0,L))$. Let $j=qK+r$ where $0 \le r < K$ and q,r are integers. Then $Path_p(\mathbf{x}'+j\cdot(0,0,L)) = Path_p(\mathbf{x}'+r\cdot(0,0,L)+qK\cdot(0,0,L)) = Path_p(\mathbf{x}'+r\cdot(0,0,L))$. Since $0 \le r < K$, this is one of the K members of the equivalence class.

Recall that M′ is the union of disjoint orbits, O_x. The union of spatial equivalence classes, each one corresponding to a disjoint orbit, equals the set of **p** paths. Recall that the gain of each path, K, equals the summation of $\tau(\mathbf{x})$ for **x** taken over the corresponding orbit.

$$K = \sum_{\boldsymbol{x} \in O_x} \tau(\boldsymbol{x}) .$$

From our lemma, if K is positive, there are K forward paths in the spatial equivalence class. If K is negative, there are −K backward paths in the spatial equivalence class.

The summation of $\tau(\mathbf{x})$ for **x** in orbits whose corresponding paths each have positive gain is the number of forward **p** paths. Likewise, the summation of $(-\tau(\mathbf{x}))$ for **x** in orbits whose corresponding paths each have negative gain is the number of backward **p** paths.

$$Let\ O_+ \equiv \{\, O_x : \; Path_p(\boldsymbol{x}') \text{ corresponds to } O_x \text{ and has } K > 0 \,\}$$

$$Let\ O_- \equiv \{\, O_x : \; Path_p(\boldsymbol{x}') \text{ corresponds to } O_x \text{ and has } K < 0 \,\}$$

The union of O_+ and O_- is M′. The summation of $\tau(\mathbf{x})$ for **x** in M′ is therefore

$$N_+ = \sum_{\boldsymbol{x}\in O_-} \tau(\boldsymbol{x}) \qquad N_- = -\sum_{\boldsymbol{x}\in O_-} \tau(\boldsymbol{x})$$

equal to the number of forward paths minus the number of backward paths.

$$N_+ - N_- = \sum_{\boldsymbol{x}\in M'} \tau(\boldsymbol{x}) .$$

Theorem 13

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{r}(\boldsymbol{x}) = (N_+ - N_-)\cdot(0,0,L) .$$

Proof

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{r}(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in M'} (\boldsymbol{p}(\boldsymbol{x}) - \boldsymbol{x}) = \sum_{\boldsymbol{x}\in M'} (\boldsymbol{p}'(\boldsymbol{x}) + \tau(\boldsymbol{x})\cdot(0,0,L) - \boldsymbol{x})$$

$$= \sum_{\boldsymbol{x}\in M'} \boldsymbol{p}'(\boldsymbol{x}) - \sum_{\boldsymbol{x}\in M'} \boldsymbol{x} + \sum_{\boldsymbol{x}\in M'} \tau(\boldsymbol{x})\cdot(0,0,L) = (N_+ - N_-)\cdot(0,0,L) .$$

Notice that if there are no throws backwards in time, there can be no backward paths and the equations becomes

$$\sum_{\boldsymbol{x}\in M'} \boldsymbol{r}(\boldsymbol{x}) = (0,0,N_+L) .$$

Looking back to theorem 6 we see that $N_+ = N$. Recall that when the concept of a state was introduced, no throws backward in time were allowed. We know that this means there cannot be any backward paths so our definition for N gave the number of forward paths. Theorem 13 is more general. Although mathematically interesting, it is not useful to ordinary jugglers, few of whom can actually throw a ball into the past!! This theorem brings up some interesting questions. Suppose we allow throwing balls into the past as well as into the future. What do N_+ and N_- really represent in terms of numbers of balls being juggled? If a person naturally progressed backward in time, he would throw all balls backward in time. In his view, N_- would represent the number of real balls while N_+ would always be zero. According to Allen Knutson, the constant $(N_+ - N_-)$ represents the number of real balls minus the number of antiballs seen in the pattern at any given moment. This is a very plausible idea. It is no more absurd than the concept of "antimatter", which can be thought of as matter moving backward in time. Let us entertain the question further. Suppose at frame 1 we throw a ball into the future so it lands at frame 2. At this time, we throw the ball back into the past exactly one frame so it lands at frame 1 where it started. Now suppose we observe the pattern. What do we see? At frame 0 nothing has happened so we see nothing. At frame 1, we see two balls suddenly appear. One is a real ball but the other is an antiball. At frame 2 they vanish! This violates our intuitive sense of continuity. We started with no ball; from frame 1 until frame 2 we had two balls; and thereafter we had none again. It is not just the number of balls that must remain constant. It is the difference between real and antiballs which does not change. We see that a ball can be created from nothing if at the same time and place, an antiball is created.

Definitions:
Let t be some real number representing a point in time. Let a ball be thrown from (m_1,h_1,t_1) to (m_2,h_2,t_2), which are points in multiplex space. At time t, the ball is *real* if $t_1 \le t < t_2$. The ball is called an *antiball* if $t_2 \le t < t_1$.

Theorem 14
Considering only the throws in a path, $Path_p(\mathbf{x})$, the number of real balls minus antiballs seen at any time t is equal to either 1 if that path is forward or –1 if that path is backward.

Proof
Consider a forward path, $Path_p(\mathbf{x})$. For each an element of the path, $\mathbf{p}^j(\mathbf{x})$, denote its time component by t_j. Let t be some point in time. Because the path is forward, there exists some integer, A, such that for $j \ge A$, $t_j > t$ and there exists some integer, B, such that for $j \le B$, $t_j < t$. If $j \ge A$, balls thrown from $\mathbf{p}^j(\mathbf{x})$ to $\mathbf{p}^{j+1}(\mathbf{x})$ cannot be real balls nor antiballs. Likewise, if $j \le B$, balls thrown from $\mathbf{p}^{j-1}(\mathbf{x})$ to $\mathbf{p}^j(\mathbf{x})$ cannot be real balls nor antiballs. There are exactly (A–B) throws remaining under consideration which take $\mathbf{p}^B(\mathbf{x})$ to $\mathbf{p}^A(\mathbf{x})$. There must be exactly one more of the real balls (N_{real}) than there are of the antiballs (N_{anti}) because of the parity involved in the right and left crossing of time t. This is easily proved using induction. Let C and D be integers such that D–C=1 and consider only the one throw from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^D(\mathbf{x})$. By definition, $N_{real}-N_{anti}$ is 1 if and only if $t_C \le t < t_D$. $N_{real}-N_{anti}$ is –1 if and only if $t_D \le t < t_C$. Furthermore, t_C and t_D are on the same side of t if and only if $N_{real}-N_{anti}$ is zero. Suppose these three statements hold for D–C=K. We show these hold for the case D–C=K+1. This implies D–(C+1)=K.

Case 1. Suppose $t_C \le t < t_D$. If $t_{C+1} > t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is real. Since t_{C+1} and t_D are on the same side of t, $N_{real}-N_{anti}$ is zero considering only these throws. When all throws are considered, N_{real} increases by 1 so $N_{real}-N_{anti}$ is 1. If $t_{C+1} \le t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is neither a real nor an antiball. Since $t_{C+1} \le t < t_D$, $N_{real}-N_{anti}$ is still 1 when all throws are considered.

Case 2. Suppose $t_D \le t < t_C$. If $t_{C+1} \le t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is an antiball. Since t_{C+1} and t_D are on the same side of t, $N_{real}-N_{anti}$ is zero considering only these throws. When all throws are considered, N_{anti} increases by 1 so $N_{real}-N_{anti}$ is –1. If $t_{C+1} > t$ then the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is neither a real nor an antiball. Since $t_D \le t < t_{C+1}$, $N_{real}-N_{anti}$ is still –1 when all throws are considered.

Case 3. Now suppose t_C and t_D are both less than or equal to t. If $t_{C+1} \le t$ then no real balls or antiballs are found so $N_{real}-N_{anti}$ remains zero. If $t_{C+1} > t$

we find the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is real. $N_{real}-N_{anti}$ is -1 considering only the other throws. Therefore when all throws are considered, $N_{real}-N_{anti}$ is zero. On the other hand, suppose t_C and t_D are both greater than t. If $t_{C+1}>t$ then no real balls or antiballs are found so $N_{real}-N_{anti}$ remains zero. If $t_{C+1}\leq t$ we find the ball thrown from $\mathbf{p}^C(\mathbf{x})$ to $\mathbf{p}^{C+1}(\mathbf{x})$ is an antiball. $N_{real}-N_{anti}$ is 1 considering only the other throws. $N_{real}-N_{anti}$ is zero when all throws are considered.

The case for a backward **p** path is proved in a similar fashion.

Theorem 15
For any periodic pattern, **p**, and any time t, the number of real balls minus the number of antiballs is equal to $(N_+ - N_-)$.

Proof
The contribution of each forward path to $N_{real}-N_{anti}$ as seen at some time t is 1. The contribution of each backward path to it is -1. Therefore the collection of forward paths (N_+) and backward paths (N_-) contributes (N_+-N_-) to $N_{real}-N_{anti}$. Since these represent a partition of multiplex space, there can be no other contributions to it so N_+-N_- must equal $N_{real}-N_{anti}$.

Extension of the Mathematics to Include Any Pattern

Although the mathematics presented this far is able to describe a large class of juggling patterns, it does not describe all of them. For example, patterns which require the juggler to throw in unusual rythms are not described by the mathematics if the time intervals between throws cannot be represented by integers multiples of some frametime. At the beginning, we stipulated that a pattern consisted of catches and throws which had to occur at discrete points in pattern space. Pattern space was then extended to multiplex space so that the pattern would be a permutation. In reality, time is continuous-- not discretized. All throws and catches in a pattern take place in continuous multiplex space. It is the set of times at which these throws and catches occur which is discrete. We already have developed many theorems that deal with multiplex space. As we shall see, it is not necessary to redevelop these for our new discretized space.

Definition of a Pattern
A pattern, $\mathbf{p}_a$, is a self-mapping of an unbounded countable subset, M_d, of continuous multiplex space (M_c).

Theorem 16
$(\mathbf{p}_a:M_d\rightarrow M_d,\ )$ forms a group which is isomorphic to $(\mathbf{p}:M\rightarrow M,\ )$.

Proof
Choose some element of M_d and call it t_0. Label all elements by t_j for some integer j so that their order is reflected by their subscripts. (i.e. $t_j>t_k$ implies $j>k$). Let **f** map M to M_d by $\mathbf{f}(m,h,j) = (m,h,t_j)$. Note that **f** is a bijection because it is both one-to-one and onto. Now let Φ map from $\mathbf{p}_a$ to **p**: $\mathbf{p} = \Phi(\mathbf{p}_a) = \mathbf{f}^{-1}(\mathbf{p}_a(\mathbf{f}))$. Φ is well-defined because **f** is a bijection. Furthermore, Φ is a bijection and we show $\Phi(\mathbf{p}_a\ \mathbf{q}_a) = \Phi(\mathbf{p}_a)\ \Phi(\mathbf{q}_a)$.

$$\Phi(\mathbf{p}_a\ \mathbf{q}_a) = \mathbf{f}^{-1}\ \mathbf{p}_a\ \mathbf{q}_a\ \mathbf{f} = \mathbf{f}^{-1}\ \mathbf{p}_a\ \mathbf{f}\ \mathbf{f}^{-1}\ \mathbf{q}_a\ \mathbf{f} = \Phi(\mathbf{p}_a)\ \Phi(\mathbf{q}_a).$$

What does this mean? Every definition and theorem before was based on **p** and the operation of composition. We have proved this new group, which encompasses all patterns, is isomorphic to our old group. Thus, to analyze any pattern, $\mathbf{p}_a$, we look at $\mathbf{p} = \Phi(\mathbf{p}_a)$. To notate a periodic $\mathbf{p}_a$, it suffices to use MHN in conjunction with the set, $\{t_1, t_2, ..\ t_L\}$.

As a side note, it should be mentioned that aperiodic patterns can be viewed as a concatenation of periodic patterns. To link pattern X to pattern Y requires that they have one state in common and the link is made at this state. Thus, the mathematics is capable of describing aperiodic patterns as well as periodic patterns.

A Brief Summary with Conclusions

The axiom of continuity provided a test for pattern validity. It was found that a pattern was represented by a permutation, **p**, on multiplex space (M), an infinite space. Periodicity and restricted multiplex space (M′) were defined and it was determined that for **x** in M′, **p**(**x**) was the sum of **p′**(**x**) and an integer multiple, $\tau(\mathbf{x})$ of (0,0,L) where **p′** was a permutation on M′ (see theorem 2). A way of notating **p** by means of a matrix of vectors was introduced called "Multi-Hand Notation" or "MHN". The operations of permutation, local translation, and global translation were found to preserve validity. The concept of states and transitions was presented and a useful theorem was proved which related the pattern, its period, and the number of balls being juggled (see theorem 6). It was shown that the entire class of patterns together with the operation of composition formed a group. Multiplex space (M) was partitioned into ball paths. It was shown that each path in M corresponded to an orbit, O_x in M′, and also to a gain, K. K was found to equal the summation of $\tau(\mathbf{x})$ for **x** in O_x. Paths which corresponded to the same orbit in M′ were termed spatially equivalent and M was partitioned into spatial equivalence classes. It was proved that each of these spatial equivalence classes had K forward paths if K was positive or −K backward paths if K was negative. The difference between the numbers of forward and backward paths, $(N_+ - N_-)$, was found to be the summation of K for all equivalence classes which was equal to the summation of $\tau(\mathbf{x})$ for **x** in M′ (see theorem 12). A theorem more general than theorem 6 was proved which related a pattern, its period, and the constant, $(N_+ - N_-)$. Antiballs and real balls were defined in terms of being visible at a specific time t. It was determined that the number of real balls minus the number of antiballs was equal to $(N_+ - N_-)$. Finally, a broader definition for a pattern was given which included arhythmic throws. It was determined that any pattern whatsoever could be identified with a rhythmic pattern through an isomorphism, Φ.

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